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Which wheel graphs are determined by their Laplacian spectra?

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ABSTRACT

The wheel graph, denoted by W_{n+1}, is the graph obtained from the circuit C_n with n vertices by adding a new vertex and joining it to every vertex of C_n. In this paper, the wheel graph W_{n+1}, except for W_7, is proved to be determined by its Laplacian spectrum, and a graph cospectral with the wheel graph W_7 is given.

1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. All graphs considered here are simple and undirected. Let matrix $A(G)$ be the (0,1)-*adjacency matrix* of G and d_k the degree of the vertex v_k. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, where $D(G)$ is the $n \times n$ diagonal matrix with $\{d_1, d_2, \ldots, d_n\}$ as diagonal entries (and all other entries 0). The polynomial $P_{L(G)}(\mu) = \det(\mu I - L(G))$, where I is the identity matrix, is called the *Laplacian characteristic polynomial* of G, which can be written as $P_{L(G)}(\mu) = q_0\mu^n + q_1\mu^{n-1} + \cdots + q_n$. Since the matrix $L(G)$ is real and symmetric, its eigenvalues, i.e., all roots of $P_{L(G)}(\mu)$, are real numbers, and are called the Laplacian eigenvalues of G. Assume that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n (= 0)$ are these eigenvalues; they compose the *Laplacian spectrum* of G. Two non-isomorphic graphs are said to be *cospectral* with respect to the Laplacian spectrum if they share the same Laplacian spectrum [1]. In the following, we call two graphs *cospectral* if they are cospectral with respect to the Laplacian spectrum.

Take two disjoint graphs G_1 and G_2. A graph G is called the *disjoint union* (or *sum*) of G_1 and G_2, denoted as $G = G_1 + G_2$, if $V(G) = V(G_1) \bigcup V(G_2)$ and $E(G) = E(G_1) \bigcup E(G_2)$. Similarly, the *product* $G_1 \times G_2$ denotes the graph obtained from $G_1 + G_2$ by adding all the edges (a, b) with $a \in V(G_1)$ and $b \in V(G_2)$. In particular, if G_2 consists of a single vertex b, we write $G_1 + b$ and $G_1 \times b$ instead of $G_1 + G_2$ and $G_1 \times G_2$, respectively. In these cases, b is called an *isolated vertex* and a *universal vertex*, respectively. A *subgraph* [1] of graph G is constructed by taking a subset S of $E(G)$ together with all vertices incident in G with some edge belonging to S. Clearly, the product graph $G_1 \times G_2$ has a complete bipartite subgraph $K_{m,n}$, where m and n are the order of G_1 and G_2, respectively.

Which graphs are determined by their spectra seems to be a difficult problem in the theory of graph spectra. Up to now, many graphs have been proved to be determined by their spectra [2–8]. In [3], the so-called *multi-fan graph* is constructed and proved to be determined by its Laplacian spectrum. Then, take the definition of the so-called *multi-wheel graph*: The multi-wheel graph is the graph $(C_{n_1} + C_{n_2} + \cdots + C_{n_k}) \times b$, where $C_{n_1} + C_{n_2} + \cdots + C_{n_k}$ is the disjoint union of circuits C_{n_i}, and $k \geq 1$ and $n_i \geq 3$ for $i = 1, 2, \ldots, k$. Note that the particular case of $k = 1$ in the definition is just the wheel graph

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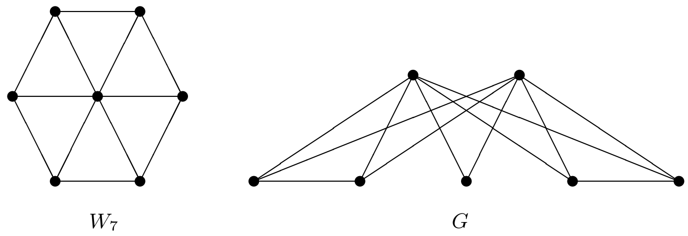

Fig. 1. The cospectral graphs W_7 and G.

$W_{n_1+1} = C_{n_1} \times b$ with $n_1 + 1$ vertices. In this paper, the wheel graph W_{n+1}, except for W_7, will be proved to be determined by its Laplacian spectrum. This method is also useful in proving that the multi-wheel graph $(C_{n_1} + C_{n_2} + \cdots + C_{n_k}) \times b$ is determined by its Laplacian spectrum, where $k \geq 2$. Here, we will skip the details of the proof for multi-wheel graphs. In [9], a new method (see Proposition 4 in [9]) is pointed out, which can be used to prove that every multi-wheel graph $(C_{n_1} + C_{n_2} + \cdots + C_{n_k}) \times b$ is determined by its Laplacian spectrum, where $k \geq 2$. But, for the wheel graph W_{n+1}, the new method in [9] is useless.

2. Preliminaries

Some previously established results about the spectrum are summarized in this section. They will play an important role throughout the paper.

Lemma 2.1 ([10]). *Let G_1 and G_2 be graphs on disjoint sets of r and s vertices, respectively. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r (= 0)$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_s (= 0)$ are the Laplacian spectra of graphs G_1 and G_2, respectively, then $r + s$; $\mu_1 + s, \mu_2 + s, \ldots, \mu_{r-1} + s$; $\eta_1 + r, \eta_2 + r, \ldots, \eta_{s-1} + r$; and 0 are the Laplacian spectra of graph $G_1 \times G_2$.*

Lemma 2.2 ([11]).

(1) *Let G be a graph with n vertices and m edges and $d_1 \geq d_2 \geq \cdots \geq d_n$ its non-increasing degree sequence. Then some of the coefficients in $P_{L(G)}(\mu)$ are*

$$q_0 = 1; \quad q_1 = -2m; \quad q_2 = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2;$$

$$q_{n-1} = (-1)^{n-1} nS(G); \quad q_n = 0$$

where $S(G)$ is the number of spanning trees in G.

(2) *For the Laplacian matrix of a graph, the number of components is determined from its spectrum.*

Lemma 2.3 ([12]). *Let graph G be a connected graph with $n \geq 3$ vertices. Then $d_2 \leq \mu_2$.*

Lemma 2.4 ([13,11]). *Let G be a graph with $n \geq 2$ vertices. Then $d_1 + 1 \leq \mu_1 \leq d_1 + d_2$.*

Lemma 2.5 ([14]). *If G is a simple graph with n vertices, then $m_G(n) \leq \lfloor \frac{d_n}{n-d_1} \rfloor$, where $m_G(n)$ is the multiplicity of the eigenvalue n of $L(G)$ and $\lfloor x \rfloor$ the greatest integer less than or equal to x.*

Lemma 2.6 ([15]). *Let $\overline{G}$ be the complement of a graph G. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $\overline{\mu}_1 \geq \overline{\mu}_2 \geq \cdots \geq \overline{\mu}_n = 0$ be the Laplacian spectra of graphs G and $\overline{G}$, respectively. Then $\mu_i + \overline{\mu}_{n-i} = n$ for any $i \in \{1, 2, \ldots, n-1\}$.*

Lemma 2.7 ([16]). *Let G be a connected graph on n vertices. Then n is an eigenvalue of Laplacian matrix $L(G)$ if and only if G is the product of two graphs.*

3. Main results

First, let us check that the graphs G and W_7 in Fig. 1 are cospectral. By using Maple, the Laplacian characteristic polynomials of the graphs G and W_7 are both

$$\mu^7 - 24\mu^6 + 231\mu^5 - 1140\mu^4 + 3036\mu^3 - 4128\mu^2 + 2240\mu.$$

That is, G and W_7 are cospectral. Then, we will have the following proposition.

Proposition 3.1. *The wheel graph W_7 is not determined by its Laplacian spectrum.*

Theorem 3.2. *The wheel graph W_{n+1}, except for W_7, is determined by its Laplacian spectrum.*

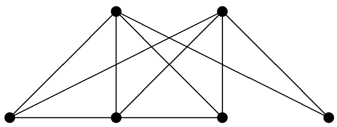

Fig. 2. Graph with the degree sequence 4, 4, 4, 3, 3, 2.

Proof. Since the Laplacian spectrum of the circuit C_n is $2-2\cos\frac{2\pi i}{n}$ $(i = 1, 2, \dots, n)$, by Lemma 2.1, the Laplacian spectrum of W_{n+1} is $3-2\cos\frac{2\pi i}{n}$ $(i = 1, 2, \dots, n-1)$, and also 0 and $n+1$. Suppose a graph G is cospectral with W_{n+1}. Lemma 2.2 implies that graph G has $n+1$ vertices, $2n$ edges and one component. Then, by Lemma 2.7, G is a product of two graphs. Let $d_1 \geq d_2 \geq \cdots \geq d_{n+1}$ be the non-increasing degree sequence of graphs G. By Lemma 2.3, $d_2 \leq \mu_2 \leq 5$, i.e., $d_2 \leq 5$. Lemma 2.4 implies that $d_1 + 1 \leq n + 1 \leq d_1 + d_2 \leq d_1 + 5$, i.e., $n - 4 \leq d_1 \leq n$. Consider the following cases for d_1.

Case 1. $d_1 = n - 4$. Since the multiplicity of the $\mu_1 = n+1$ is 1, by Lemma 2.5, $1 \leq \lfloor \frac{d_{n+1}}{n+1-(n-4)} \rfloor$, i.e., $d_{n+1} \geq 5$. Then, $d_2 = d_3 = \cdots = d_n = d_{n+1} = 5$, i.e., there exist at least n vertices of degree five in graph G. But, $5n + (n-4) \neq 2(2n)$, a contradiction to $\sum_{i=1}^{n+1} d_i = 2m$, where m is the number of edges in G.

Case 2. $d_1 = n - 3$. Since the multiplicity of the $\mu_1 = n+1$ is 1, by Lemma 2.5, $1 \leq \lfloor \frac{d_{n+1}}{n+1-(n-3)} \rfloor$, i.e., $d_{n+1} \geq 4$. Except for the vertex of degree $d_1 = n-3$, suppose there still exist x_5 vertices of degree five and x_4 vertices of degree four in graph G. $\sum_{i=1}^{n+1} d_i = 2m$ implies the following equations:

$$\begin{cases} x_5 + x_4 + 1 = n + 1 \\ 5x_5 + 4x_4 + (n-3) = 2 \times 2n. \end{cases}$$

Clearly, $x_5 = 3 - n, x_4 = 2n - 3$ is the solution of the equations. But $x_5 < 0$, a contradiction.

Case 3. $d_1 = n - 2$. By Lemma 2.5, $1 \leq \lfloor \frac{d_{n+1}}{n+1-(n-2)} \rfloor$, i.e., $d_{n+1} \geq 3$. Except for the vertex of degree $d_1 = n-2$, suppose there still exist x_5 vertices of degree five, x_4 vertices of degree four and x_3 vertices of degree three in G. Lemma 2.2 and $\sum_{i=1}^{n+1} d_i = 2m$ imply the following equations:

$$\begin{cases} x_5 + x_4 + x_3 + 1 = n + 1 \\ 5x_5 + 4x_4 + 3x_3 + (n-2) = 2 \times 2n \\ 25x_5 + 16x_4 + 9x_3 + (n-2)^2 = n^2 + 9n. \end{cases}$$

Clearly, $x_5 = 2n - 9, x_4 = 20 - 4n, x_3 = 3n - 11$. For $n = 4, x_5 < 0$, a contradiction. For $n = 5, x_5 = 1$, but $d_1 = 3 < 5$, a contradiction. For $n \geq 7, x_4 < 0$, a contradiction.

Case 4. $d_1 = n - 1$. By Lemma 2.5, $1 \leq \lfloor \frac{d_{n+1}}{n+1-(n-1)} \rfloor$, i.e., $d_{n+1} \geq 2$. Except for the vertex of degree $d_1 = n-1$, suppose that there still exist x_5 vertices of degree five, x_4 vertices of degree four, x_3 vertices of degree three and x_2 vertices of degree two in graph G. Lemma 2.2 and $\sum_{i=1}^{n+1} d_i = 2m$ imply the following equations:

$$\begin{cases} x_5 + x_4 + x_3 + x_2 + 1 = n + 1 \\ 5x_5 + 4x_4 + 3x_3 + 2x_2 + (n-1) = 2 \times 2n \\ 25x_5 + 16x_4 + 9x_3 + 4x_2 + (n-1)^2 = n^2 + 9n. \end{cases}$$

By solving these equations, $x_4 = n - 3 - 3x_5, x_3 = 7 - n + 3x_5, x_2 = n - 4 - x_5$, where x_5 is an integer. And $x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$ imply that $\max\{\frac{n-7}{3}, 0\} \leq x_5 \leq \min\{\frac{n-3}{3}, n-4\}$. Clearly, $\frac{n-3}{3} < n - 4$ for $n \geq 5$. Therefore, if $n \geq 5$, then $x_2 > 0$, i.e., there must exist vertices of degree two in graph G. Note that G is a product of two graphs and G has a complete bipartite subgraph K_{m_1,m_2}, where $m_1 + m_2 = n + 1$. Then, for $n \geq 5$, the existence of vertices with degree two implies that the complete bipartite subgraph K_{m_1,m_2} is $K_{n-1,2}$ or $K_{n,1}$. But for $K_{m_1,m_2} = K_{n,1}$, there will exist a vertex with degree n in graph G, a contradiction to $d_1 = n - 1$. For $n \geq 7$, $K_{n-1,2}$ implies that there at least exist two vertices with degree no less than $n - 1$, a contradiction. Consider the following cases for x_5 and $n \leq 5$.

Case 4.1. $x_5 = 0$. Clearly, $x_4 = n - 3, x_3 = 7 - n, x_2 = n - 4$. Consider the following cases.

Case 4.1.1. $n = 3$. Clearly, $x_2 = -1 < 0$, a contradiction.

Case 4.1.2. $n = 4$. Clearly, $d_1 = 3, x_4 = 1, x_3 = 3, x_2 = 0$, but $d_1 = 3 < 4$, a contradiction.

Case 4.1.3. $n = 5$. Clearly, $d_1 = 4, x_4 = 2, x_3 = 2, x_2 = 1$, i.e., there exist three vertices of degree four, two vertices of degree three and one vertex of degree two in graph G. All the graphs with three vertices of degree four, two vertices of degree three and one vertex of degree two and with complete bipartite subgraph $K_{2,4}$ have been enumerated; they are isomorphic to the graph shown in Fig. 2. By using Maple, the Laplacian characteristic polynomials of the graphs G and W_6 are

$$P_{L(G)}(\mu) = \mu^6 - 20\mu^5 + 155\mu^4 - 580\mu^3 + 1044\mu^2 - 720\mu,$$

$$P_{L(W_6)}(\mu) = \mu^6 - 20\mu^5 + 155\mu^4 - 580\mu^3 + 1045\mu^2 - 726\mu.$$

Clearly, they have different Laplacian characteristic polynomials, a contradiction.

Case 4.2. $x_5 \geq 1$. Clearly, for $3 \leq n \leq 5$, $x_4 = n - 3 - 3x_5 < 0$, a contradiction.

Case 5. $d_1 = n$. Since both G and W_{n+1} have the largest degree n, $\overline{W_{n+1}} = \overline{C_n} + b$ and $\overline{G} = \overline{G'} + b$, where $\overline{G'}$ is an unknown graph. Lemma 2.6 implies that $\overline{G}$ and $\overline{W_{n+1}}$ are cospectral, i.e., $\overline{C_n}$ and $\overline{G'}$ are cospectral. Since the circuit C_n is determined by its Laplacian spectrum [6], so is its complement $\overline{C_n}$. Then, $\overline{G'}$ is isomorphic to $\overline{C_n}$, i.e., $\overline{G}$ is isomorphic to $\overline{W_{n+1}}$. Therefore G is isomorphic to W_{n+1}. □

For a graph, its Laplacian eigenvalues determine the eigenvalues of its complement [15], so the complements of all the wheel graphs W_{n+1}, except for W_7, are determined by their Laplacian spectra.

4. Conclusion

In this paper, the wheel graph W_{n+1}, except for W_7, is proved to be determined by its Laplacian spectrum by showing that a graph G cospectral to the wheel graph W_{n+1} must have a universal vertex, and this is the key point of the paper.

We would like to close this paper by posing an interesting question. Since the wheel graph $W_{n+1} = C_n \times b$ for $n \neq 6$ and the fan graph $F_{n+1} = P_n \times b$ (see [3]) are proved to be determined by their Laplacian spectrum, C_n and P_n are also determined by their Laplacian spectrum (see [6]); our question is that which graphs satisfy the following relation:

"*If G is a graph determined by its Laplacian spectrum, then G × b is also determined by its Laplacian spectrum.*"

If G is disconnected, i.e., G has at least two components, then the above relation is true (see Proposition 4 in [9]). But, if G is connected, it is known that only the complete graph K_n, the circuit C_n with $n \neq 6$ and the path P_n satisfy the above relation until now.

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